TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 325, Number 1, May 1991

SUBELLIPTIC ESTIMATES FOR THE $\bar{\partial}$-NEUMANN PROBLEM FOR $n-1$ FORMS

LOP-HING HO

ABSTRACT. In this note we deal with the problem of the subelliptic estimates of the $\bar{\partial}$-Neumann problem on nonpseudoconvex domains. In the first part we give a necessary condition for $n-1$ forms in a class of domains. In the second part we give the exact estimate for a class of domains where the Levi form of a vector field L is bounded below by a certain function.

1. INTRODUCTION

Let Ω be a domain in $\mathbb{C}^n$. We say that a subelliptic estimate of order ε holds at $x_0 \in \overline{\Omega}$ for (p, q) forms if there is a neighborhood U of x_0, $C > 0$, $0 < \varepsilon < 1$ such that

$$|||u|||_\varepsilon^2 \le C(\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 + \|u\|^2)$$

for all (p, q) forms $u \in \mathrm{Dom}(\bar{\partial}^*)$ with coefficients supported in $U \cap \overline{\Omega}$.

The existence of subelliptic estimates has important applications in the boundary regularity of solutions of $(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})u = f$. A lot of work has been done on subelliptic estimates for pseudoconvex domains. (See [1, 2, 5 and 6].) On nonpseudoconvex domains the case $\varepsilon = \frac{1}{2}$ is completely settled. (See [3, 4, and 5].) For $\varepsilon < \frac{1}{2}$ we proved in [4] that if there is a holomorphic vector field L whose Levi-form is nonnegative and L is finite type at x_0, then there is a subelliptic estimate at x_0 for $n-1$ forms.

Now consider the domain defined by

$$\Omega = \{r(z) < 0 \colon r(z) = 2\,\mathrm{Re}\, z_3 - |z_1 z_2|^2 + |z_2|^6\}.$$

The vector field $L = z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} + 3|z_2|^6 \frac{\partial}{\partial z_3}$ (which degenerates at the origin) has a nonnegative Levi form near $z = 0$. We will prove in Theorem 2.1 that this domain does not have a subelliptic estimate for 2 forms at $z = 0$. In §3 we prove that in some domains if L is of type m at x_0, then a subelliptic estimate of order $\varepsilon = \frac{1}{m}$ holds at x_0 for $n-1$ forms.

The author would like to thank D. Catlin for helpful suggestions.

Received by the editors September 23, 1988 and, in revised form, April 13, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35N15.

2. Necessary conditions for subelliptic estimates for $n-1$ forms

Theorem 2.1. *Let* $\Omega = \{z \in \mathbb{C}^n : r(z) < 0, r \in C^\infty$ *near the origin*$\}$ *be a domain in* $\mathbb{C}^n$, *and there are positive integers* p, $1 \le p \le n-1$, m *and* m_k, $k = p+1, \ldots, n-1$ *such that*

(i) $r(z) = 2\operatorname{Re} z_n + O(|z|^2)$,

(ii) *if* $|\alpha + \beta| \ge 2$, $\alpha_n = \beta_n = 0$ *and*

$$\sum_{k=1}^{p} \frac{\alpha_k + \beta_k}{m} + \sum_{k=p+1}^{n-1} \frac{\alpha_k + \beta_k}{m_k} < 1$$

then $D_z^\alpha D_{\bar z}^\beta r(0) = 0$.

(iii) *Set* $\rho(z_{p+1}, \ldots, z_{n-1}) = \sum_{\alpha,\beta} D_z^\alpha D_{\bar z}^\beta r(0) \frac{z^\alpha \bar z^\beta}{\alpha! \beta!}$ *where the sum is taken over* (α, β) *with* $\alpha_j = \beta_j = 0$ *for* $j = 1, \ldots, p, n$ *and with*

$$\sum_{k=p+1}^{n-1} \frac{\alpha_k + \beta_k}{m_k} = 1.$$

Then there is a positive constant C *such that*

$$\rho(z_{p+1}, \ldots, z_{n-1}) \le -C \sum_{k=p+1}^{n-1} |z_k|^{m_k}.$$

Then if a subelliptic estimate of order ε *holds for* $n-1$ *forms near the origin, then* $\varepsilon \le \frac{1}{m}$.

Denote $z' = (z_1, \ldots, z_p)$, $z'' = (z_{p+1}, \ldots, z_{n-1})$. To prove the theorem, we need the following lemma.

Lemma 2.2. *With the same assumptions as in Theorem* 2.1, *there is a polynomial* ρ_1 *and a function* ρ_2 *smooth near the origin such that*

$$r(z) = 2\operatorname{Re} z_n + \rho(z'') + \rho_1(z', z'') + \rho_2(z) \tag{2.1}$$

with the following properties:

(a) *Given any* $\varepsilon > 0$, *there is a* $\varphi \in C_0^\infty(\mathbb{R})$ *such that the following holds for all* $t > 0$ *when* $(z', z'') \in \operatorname{supp}\{\varphi(t^{\frac{1}{m}} x_1)\varphi(t^{\frac{1}{m}} y_1) \cdots \varphi(t^{\frac{1}{m}} y_p)\varphi(x_{p+1}) \cdots \varphi(y_{n-1})\}$.

(i) $|\rho_1(z', z'')| \le \varepsilon(-\rho(z'') + \frac{1}{t})$,

(ii) $|\partial \rho_1 / \partial z_1(z', z'')| \le \varepsilon t^{\frac{1}{m}}(-\rho(z'') + \frac{1}{t})$,

(iii) $|\partial \rho_2 / \partial z_1| \le \varepsilon(|z_n| + |\rho| + \frac{1}{t})$.

(b) *There is a smooth function* χ *such that*

(i) $\operatorname{Re} \chi = \frac{1}{2}\rho_2$,

(ii) $\frac{\partial \chi}{\partial \bar z_n} = O(|z_n|)$.

Proof. Let $M = \max_{p+1\le i\le n-1}\{m, m_i\}$. We expand the Taylor series of $r(z)$ up to order M. Let

$$\rho_1(z', z'') = \sum_{\substack{|\alpha|+|\beta|\le M\\ \alpha_n+\beta_n=0}} D_z^\alpha D_{\bar z}^\beta r(0)\frac{z^\alpha \bar z^\beta}{\alpha!\beta!} - \rho(z'')$$

and $\rho_2(z) = r - 2\operatorname{Re} z_n - \rho - \rho_1$.

Then we have

$$r = 2\operatorname{Re} z_n + \rho + \rho_1 + \rho_2$$

as required.

To prove (a) (i), let $z^\alpha \bar z^\beta$ be a term in ρ_1. We separate the proof into two cases:

Case 1. $|\alpha' + \beta'| = 0$.

In this case we must have $\sum_{k=p+1}^{n-1}(\alpha_k + \beta_k)/m_k > 1$. Hence there are real numbers $a_k \ge 0$ with $\sum_{k=p+1}^{n-1} a_k/m_k = 1$ and $a > 0$ such that

$$\begin{aligned} |z^\alpha \bar z^\beta| &\le |z''|^a \prod_{k=p+1}^{n-1} (|z_k|^{m_k})^{a_k/m_k} \\ &\le |z''|^a \sum_{k=p+1}^{n-1} \frac{a_k}{m_k}|z_k|^{m_k} \le \varepsilon(-\rho) \end{aligned}$$

for $z'' \in \operatorname{supp}\{\varphi(x_{p+1})\varphi(y_{p+1})\cdots\varphi(y_{n-1})\}$ when $\operatorname{supp}\varphi$ is small enough. Note that we have used the inequality

$$\prod a_i^{\alpha_i} \le \sum \alpha_i a_i$$

where $a_i \ge 0$, $\alpha_i \ge 0$ and $\sum \alpha_i = 1$ in line 2 above.

Case 2. $|\alpha' + \beta'| \ne 0$.

We assume that

$$\sum_{i=1}^{p} \frac{\alpha_i + \beta_i}{m} + \sum_{i=p+1}^{n-1} \frac{\alpha_i + \beta_i}{m_i} = 1$$

since the case

$$\sum_{i=1}^{p} \frac{\alpha_i + \beta_i}{m} + \sum_{i=p+1}^{n-1} \frac{\alpha_i + \beta_i}{m_i} > 1$$

follows immediately.

Now using the inequality in Case 1 again, we have

$$\begin{aligned}|z^{\alpha}\overline{z}^{\beta}| &= \prod_{i=1}^{p} |t^{1/m} z_i|^{\alpha_i+\beta_i} \prod_{i=p+1}^{n-1} |t^{1/m_i} z_i|^{\alpha_i+\beta_i} t^{-(\sum_{i=1}^{p}(\alpha_i+\beta_i)/m + \sum_{i=p+1}^{n-1}(\alpha_i+\beta_i)/m_i)} \\ &\leq \varepsilon \left(\sum_{i=p+1}^{n-1} \frac{\alpha_i+\beta_i}{m_i} |t^{1/m_i} z_i|^{m_i} + 1 \right) t^{-1} \\ &\leq C\varepsilon \left(\sum_{i=p+1}^{n-1} |z_i|^{m_i} + \frac{1}{t} \right)\end{aligned}$$

for $z \in \operatorname{supp}\{\varphi(t^{1/m}x_1)\cdots\varphi(t^{1/m}y_p)\varphi(x_{p+1})\cdots\varphi(y_{n-1})\}$.

Combining these two cases, we see that for some $\varphi \in C_0^{\infty}(\mathbb{R})$ we have $|\rho_1(z', z'')| \leq \varepsilon(-\rho(z'') + \frac{1}{t})$.

For (a)(ii) we note that if the power in α_1 is decreased by 1, then the power of t will be increased by $\frac{1}{m}$. The same argument goes through as in (i).

To prove (a)(iii) we need to observe the fact that ρ_2 consists of terms which involve z_n or which are of order $M+1$ in (z', z''). Again we use the argument in the proof of (i) and we are done.

For (b), we note that

$$\rho_2(z) = \left.\frac{\partial \rho_2}{\partial z_n}\right|_{z_n=0} z_n + \left.\frac{\partial \rho_2}{\partial \overline{z}_n}\right|_{z_n=0} \overline{z}_n + \lambda(z)$$

where λ is a sum of terms of order 2 in z_n and order $M+1$ in (z', z''). Now we set $\chi(z) = \partial\rho_2/\partial z_n|_{z_n=0} + \frac{1}{2}\lambda(z)$, then (i) and (ii) follows immediately.

Proof of Theorem 2.1. Define vector fields

$$\begin{aligned}L_i &= r_{z_i}\frac{\partial}{\partial z_n} - r_{z_n}\frac{\partial}{\partial z_i}, \qquad 1 \leq i \leq n-1, \\ L_n &= \frac{1}{r_{z_n}}\frac{\partial}{\partial z_n}.\end{aligned}$$

Let $\omega_1, \omega_2, \ldots, \omega_n$ be $(1, 0)$ forms dual to $L_1, \ldots, L_n$. Define a sequence of $n-1$ forms

$$U_t = (z_n + \rho + \rho_1 + \chi - \tfrac{1}{t})^{-q}\Phi(z)\overline{\omega}_1 \wedge \overline{\omega}_2 \wedge \cdots \wedge \overline{\omega_{n-1}}$$

where ρ_1, χ are the same as in Lemma 2.2, q is a fixed large positive number and $\Phi(z) = \varphi(t^{1/m}x_1)\varphi(t^{1/m}y_1)\cdots\varphi(t^{1/m}y_p)\varphi(x_{p+1})\cdots\varphi(y_{n-1})\varphi(x_n)\varphi(y_n)$.

To show that U_t is well defined, we observe that

$$\begin{aligned}\operatorname{Re}\left(z_n + \rho + \rho_1 + \chi - \frac{1}{t}\right) &= \frac{r - \rho - \rho_1 - \rho_2}{2} + \rho + \rho_1 + \frac{1}{2}\rho_2 - \frac{1}{t} \\ &= \frac{r + \rho + \rho_1}{2} - \frac{1}{t} < 0\end{aligned}$$

for $z \in \operatorname{supp}\Phi$ in view of Lemma 2.2.

We will use U_t to denote the function as well as the $n-1$ form from now on. Clearly

$$\|\bar\partial U_t\|^2 + \|\bar\partial^* U_t\|^2 + \|U_t\|^2 \approx \|L_1 U_t\|^2 + \cdots + \|L_{n-1} U_t\|^2 + \|\bar L_n U_t\|^2 + \|U_t\|^2.$$

As in the proof of Theorem 3.1 in [4], we estimate the order of t in $|||U_t|||_\varepsilon^2$ and in $\|\bar\partial U_t\|^2 + \|\bar\partial^* U_t\|^2 + \|U_t\|^2$, thus forcing $\varepsilon \le \frac{1}{m}$.

Denote $\Psi = z_n + \rho + \rho + \chi - \frac{1}{t}$, then

$$L_1 U_t = -qr_{z_1}\left(1 + \frac{\partial\chi}{\partial z_n}\right)\Psi^{-q-1}\Phi + r_{z_1}\Psi^{-q}\frac{\partial\Phi}{\partial z_n} + qr_{z_n}\left(\frac{\partial\rho_1}{\partial z_1} + \frac{\partial\chi}{\partial z_1}\right)\Psi^{-q-1}\Phi - t^{1/m} r_{z_n}\Psi^{-q}\Phi^* \tag{2.2}$$

where $\Phi^* = 0$ outside $\operatorname{supp}\Phi$.

We first estimate

$$I = \int_\Omega \frac{|r_{z_1}|^2}{|z_n + \rho + \rho_1 + \chi - \frac{1}{t}|^{2q+2}}\Phi^2(z)\,dx_1\,dy_1\cdots dy_n.$$

Consider the new coordinates $(x_1, \dots, y_{n-1}, r, Y_n)$ where

$$\begin{aligned} Y_n &= \operatorname{Im}\left(z_n + \rho + \rho_1 + \chi - \frac{1}{t}\right) \\ &= y_n + \operatorname{Im}\left(\left.\frac{\partial\rho_2}{\partial z_n}\right|_{z_n=0} z_n\right). \end{aligned}$$

For $z \in \operatorname{supp}\Phi$, we have

$$\begin{aligned} \frac{\partial(x_1, \dots, y_{n-1}, r, Y_n)}{\partial(x_1, \dots, y_{n-1}, x_n, y_n)} &= \frac{\partial(r, Y_n)}{\partial(x_n, y_n)} \\ &= \begin{vmatrix} 2 + \dfrac{\partial\rho_2}{\partial x_n} & \operatorname{Im}\left.\dfrac{\partial\rho_2}{\partial z_n}\right|_{z_n=0} \\ \dfrac{\partial\rho_2}{\partial y_n} & 1 + \operatorname{Re}\left.\dfrac{\partial\rho_2}{\partial z_n}\right|_{z_n=0} \end{vmatrix} \\ &\approx 1. \end{aligned}$$

Moreover using the implicit function theorem on (2.1) we may see that

$$x_n = \frac{r + \rho + \rho_1}{2} + O(|(z', z'')|(|r| + |Y_n|)) + \text{terms of order } M+1 \text{ in } (z', z'')$$

and

$$y_n = \frac{1}{1 + \operatorname{Im}(\partial\rho_2/\partial z_n|_{z_n=0})}\left(Y_n - \left(\operatorname{Re}\left.\frac{\partial\rho_2}{\partial z_n}\right|_{z_n=0}\right)x_n\right).$$

Hence using Lemma 2.2, we get

$$|z_n| \le C(|r| + |\rho| + |Y_n| + \tfrac{1}{t}).$$

Now

$$|r_{z_1}| = \left|\frac{\partial \rho_1}{\partial z_1} + \frac{\partial \rho_2}{\partial z_1}\right| \leq Ct^{1/m}\left(|r| + |\rho| + |Y_n| + \frac{1}{t}\right)$$

using (a)(ii), (iii) of Lemma 2.2 and the above inequality.

Hence

$$I \leq Ct^{2/m}\int_\Omega \frac{(|r| + |\rho| + |Y_n| + \frac{1}{t})^2}{||r + \frac{\rho}{2} - \frac{1}{t}|^2 + Y_n^2|^q}\Phi^2(z)\,dx_1\cdots dy_{n-1}\,dr\,dy_n.$$

By a change of variables

$$\begin{aligned} \tilde{z}_i &= t^{1/m}z_i, \qquad 1 \leq i \leq p, \\ \tilde{z}_i &= t^{1/m_i}z_i, \qquad p+1 \leq i \leq n-1, \\ \tilde{r} &= tr, \quad \widetilde{Y}_n = tY_n \end{aligned}$$

we get

$$I \leq Ct^{2q-2+\frac{2-2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{\tilde{r}=-\infty}^{0}\frac{(|\tilde{r}| + \sum|\tilde{z}_i|^{m_i} + |\widetilde{Y}_n| + 1)^2}{||\tilde{r} + \frac{\rho}{2} - 1|^2 + \widetilde{Y}_n^2|^q}$$
$$\times\, \varphi^2(\tilde{x}_1)\varphi^2(\tilde{y}_1)\cdots\varphi^2(\tilde{y}_p)d\tilde{x}_1\ldots d\tilde{y}_{n-1}d\tilde{r}\,d\widetilde{Y}_n.$$

It is not hard to see that the integral on the right-hand side is finite and hence we have $I \leq Ct^{2q-2+\frac{2-2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}$.

For the third term in (2.2), we use (a)(ii) of Lemma 2.2 and the fact that

$$\frac{\partial\chi}{\partial z_1} = O(|z_n|) + \text{terms of order } M \text{ in } (z', z'')$$

we get

$$\left|\frac{\partial\rho_1}{\partial z_1} + \frac{\partial\chi}{\partial z_1}\right| \leq Ct^{1/m}\left(|r| + |\rho| + |Y_n| + \frac{1}{t}\right).$$

Again we have the L^2-norm of the third term of equation (2.2) is less than $Ct^{2q-2+\frac{2-2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}$.

For the second and the fourth term in (2.2) we only need to use the fact that $|r_{z_1}| \leq 1$ and $|r_{z_n}| \leq 1$. Thus we get

$$\|L_1U_t\|^2 \leq Ct^{2q-2+\frac{2-2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}. \tag{2.3}$$

Obviously (2.3) holds when the L_1 in the left-hand side is replaced by L_i, $i = 2, 3, \ldots, p$. Now

$$\begin{aligned} L_{p+1}U_t &= r_{z_{p+1}}\frac{\partial U_t}{\partial z_n} - r_{z_n}\frac{\partial U_t}{\partial z_{p+1}} \\ &= -qr_{z_{p+1}}\left(1 + \frac{\partial\chi}{\partial z_n}\right)\Psi^{-q-1}\Phi + r_{z_{p+1}}\Psi^{-q}\frac{\partial\Phi}{\partial z_n} \\ &\quad + qr_{z_n}\left(\frac{\partial\rho}{\partial z_{p+1}} + \frac{\partial\rho_1}{\partial z_{p+1}} + \frac{\partial\chi}{\partial z_{p+1}}\right)\Psi^{-q-1}\Phi - r_{z_n}\Psi^{-q}\frac{\partial\Phi}{\partial z_{p+1}}. \end{aligned}$$

The L^2-norms of the second and the fourth terms are estimated in the same way as above. Now we calculate the sum of the first and the third terms. Using (2.1) we get

$$
\begin{aligned}
& r_{z_{p+1}}\left(1+\frac{\partial\chi}{\partial z_n}\right) - r_{z_n}\left(\frac{\partial\rho}{\partial z_{p+1}}+\frac{\partial\rho_1}{\partial z_{p+1}}+\frac{\partial\chi}{\partial z_{p+1}}\right) \\
&= \left(\frac{\partial\rho}{\partial z_{p+1}}+\frac{\partial\rho_1}{\partial z_{p+1}}+\frac{\partial\rho_2}{\partial z_{p+1}}\right)\left(1+\frac{\partial\chi}{\partial z_n}\right) \\
&\quad - \left(1+\frac{\partial\rho_2}{\partial z_n}\right)\left(\frac{\partial\rho}{\partial z_{p+1}}+\frac{\partial\rho_1}{\partial z_{p+1}}+\frac{\partial\chi}{\partial z_{p+1}}\right) \\
&= \left(\frac{\partial\rho}{\partial z_{p+1}}+\frac{\partial\rho_1}{\partial z_{p+1}}\right)\left(\frac{\partial\chi}{\partial z_n}-\frac{\partial\rho_2}{\partial z_n}\right) \\
&\quad + \frac{\partial\rho_2}{\partial z_{p+1}}\left(1+\frac{\partial\chi}{\partial z_n}\right) - \frac{\partial\chi}{\partial z_{p+1}}\left(1+\frac{\partial\rho_2}{\partial z_n}\right).
\end{aligned}
\tag{2.4}
$$

From the definition of χ in Lemma 2, we see that $\partial\chi/\partial z_n - \partial\rho_2/\partial z_n$ is the sum of terms of order 1 in z_n or order M in (z', z''). $\partial\rho_2/\partial z_{p+1}$ and $\partial\chi/\partial z_{p+1}$ are also the sums of terms of order 1 in z_n or order M in (z', z'').

Hence the absolute value of the term in (2.4) is bounded by $C(|z_n|+|\rho|+\frac{1}{t})$. We can proceed in the same way as in the computation of I and we get

$$\|L_i U_t\|^2 \le C t^{2q-2+\frac{2-2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}, \qquad i=p+1,\dots,n-1.$$

Next,

$$\overline{L}_n U_t = \frac{1}{r_{\bar z_n}}\frac{\partial U_t}{\partial \bar z_n} = \frac{-q}{r_{\bar z_n}}\left(\frac{\partial\chi}{\partial\bar z_n}\right)\Psi^{-q-1}\Phi - \frac{1}{r_{\bar z_n}}\Psi^{-q}\frac{\partial\Phi}{\partial\bar z_n}.$$

From (b) of Lemma 2.3, $\partial\chi/\partial\bar z_n = O(|z_n|)$ and it is easy to see that we get

$$\|\overline{L}_n U_t\|^2 \le C t^{2p-2+\frac{2-2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}.$$

Similarly,

$$\|U_t\|^2 \le C t^{2q-2+\frac{2-2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}.$$

Combining all these estimates we finally get

$$\|\bar\partial U_t\|^2 + \|\bar\partial^* U_t\|^2 + \|U_t\|^2 \le C t^{2q-2+\frac{2-2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}. \tag{2.5}$$

We proceed to find a lower bound for $|||U_t|||_\varepsilon^2$. We introduce coordinates $(x_1, \dots, x_{n-1}, y_{n-1}, y_n, r)$, and let $(\xi_1, \xi_2, \dots, \xi_{2n-1})$ be the Fourier transform variable dual to $(x_1, \dots, y_{n-1}, y_n)$ which are the tangential directions.

$$|||U_t|||_\varepsilon^2 = \int_{-\infty}^0 \int_{\mathbf{R}^{2n-1}} (1+|\xi|^2)^\varepsilon |\widehat{U}_t(\xi, r)|^2 \, d\xi \, dr$$
$$\geq \int |\xi_{2n-1}|^{2\varepsilon} \left| \int \frac{1}{\Psi_q} e^{-i\xi_{2n-1}y_n} \varphi(x_n)\varphi(y_n)\, dy_n \right|^2$$
$$\times (\varphi(t^{1/m}x_1)\cdots\varphi(t^{1/m}y_p)\varphi(x_{p+1})\cdots\varphi(y_{n-1}))^2 \, d\xi_{2n-1}\, dr\, dx_1 \cdots dy_{n-1}$$

where x_n is a function of $(x_1, \dots, y_{n-1}, y_n, r)$.

By a change of variables

$$\tilde{z}_i = t^{1/m} z_i, \qquad 1 \leq i \leq p,$$
$$\tilde{z}_i = t^{1/m_i} z_i, \qquad p+1 \leq i \leq n-1,$$
$$\tilde{y}_n = t y_n, \qquad \tilde{\xi}_{2n-1} = \frac{\xi_{2n-1}}{t}, \quad \text{and} \quad \tilde{r} = tr$$

we get

$$|||U_t|||_\varepsilon^2 \geq t^{2q-2+2\varepsilon-\frac{2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}} I_t$$

where

$$I_t = \int |\tilde{\xi}_{2n-1}|^{2\varepsilon} \left| \int \frac{1}{\lambda^q} e^{i\tilde{\xi}_{2n-1}\tilde{y}_n} \varphi(x_n)\varphi\left(\frac{\tilde{y}_n}{t}\right) d\tilde{y}_n \right|^2$$
$$\times (\varphi(\tilde{x}_1)\cdots\varphi(\tilde{y}_r)\varphi(t^{\overline{-m_{p+1}}}\tilde{x}_{p+1})\cdots$$
$$\varphi(t^{\overline{-m_{n-1}}}\tilde{y}_{n-1}))^2 \, d\tilde{\xi}_{2n-1}\, d\tilde{r}\, d\tilde{x}_1\, d\tilde{y}_1 \cdots d\tilde{y}_{n-1}.$$

The function λ inside the integral satisfies that for fixed $(\tilde{z}', \tilde{z}'', \tilde{\xi}_{2n-1}, \tilde{r})$, if $(\tilde{z}', \tilde{z}'') \in \operatorname{supp}\{\varphi(\tilde{x}_1)\cdots\varphi(\tilde{y}_p)\varphi(t^{\frac{1}{-m_{p+1}}}\tilde{x}_{p+1})\cdots\varphi(t^{\frac{1}{-m_{n-1}}}\tilde{y}_{n-1})\}$, then as $t \to \infty$ we have

$$\lambda(\tilde{z}', \tilde{z}'', \tilde{y}_n, \tilde{r}) \to \tfrac{\tilde{r}}{2} + \tfrac{\rho}{2} - 1 + i\tilde{y}_n + r(\tilde{z}', \tilde{z}'')$$

where $|\gamma(\tilde{z}', \tilde{z}'')| \leq \delta(-\rho+1)$ for some small δ.

Hence as $t \to \infty$ by the Dominated Convergence Theorem

$$\int \frac{1}{\lambda^q} e^{-i\tilde{\xi}_{2n-1}\tilde{y}_n} \varphi(x_n)\varphi\left(\frac{\tilde{y}_n}{t}\right) d\tilde{y}_n$$
$$\to \int \frac{1}{(\frac{\tilde{r}}{2}+\frac{\rho}{2}-1+r+i\tilde{y}_n)^q} e^{-i\tilde{\xi}_{2n-1}\tilde{y}_n}\, d\tilde{y}_n.$$

By Fatou's lemma we have

$$\liminf I_t \geq \int |\tilde{\xi}_{2n-1}|^{2\varepsilon} \left| \int \frac{1}{(\frac{\tilde{r}}{2}+\frac{\rho}{2}-1+i\tilde{y}_n+r)^q} e^{-i\tilde{\xi}_{2n-1}\tilde{y}_n}\, d\tilde{y}_n \right|^2$$
$$\times (\varphi(\tilde{x}_1)\cdots\varphi(\tilde{y}_p))^2 \, d\tilde{\xi}_{2n-1}\, d\tilde{r}\, d\tilde{x}_1\, d\tilde{y}_1 \cdots d\tilde{y}_{n-1}$$
$$\geq C > 0.$$

Hence

$$|||U_t|||_\varepsilon^2 \geq Ct^{2q-2+2\varepsilon-\frac{2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}$$

for infinitely many t's.

Combining this with (2.5), we have

$$\begin{aligned} C_1 t^{2q-2+2\varepsilon-\frac{2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}} &\leq |||U_t|||_\varepsilon^2 \leq C(\|\bar{\partial}U_t\|^2 + \|\bar{\partial}^* U_t\|^2 + \|U_t\|^2) \\ &\leq C_2 t^{2q-2+\frac{2-2p}{m}-\sum_{i=p+1}^{n-1}\frac{2}{m_i}}. \end{aligned}$$

Thus $\varepsilon \leq \frac{1}{m}$. This finishes the proof.

Corollary 2.3. *There exist domains Ω_1, Ω_2 and Ω_3 in $\mathbb{C}^3$ such that*

(i) $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$ *near the origin;*
(ii) *a subelliptic estimate for 2-forms holds for Ω_1 and Ω_3 at $z = 0$;*
(iii) *there is no subelliptic estimate for 2-forms for Ω_2.*

Proof. Let

$$\Omega_1 = \{r(z) < 0 \colon r(z) = 2\,\mathrm{Re}\, z_3 + |z_2|^4\},$$
$$\Omega_3 = \{r(z) < 0 \colon r(z) = 2\,\mathrm{Re}\, z_3 - |z_1|^2 + |z_2|^4\}.$$

Then Ω_1 and Ω_3 both have a subelliptic estimate for 2-forms at $z = 0$. Let $\Omega_2 = \{r(z) < 0 \colon r(z) = 2\,\mathrm{Re}\, z_3 - |z_1 z_2|^2 + |z_2|^4\}$. Then clearly $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$. At $(\delta, 0, 0) \in b\Omega_2$, from Theorem 2.1, there is no subelliptic estimate for 2-forms. Hence there is no subelliptic estimate for 2-forms at $z = 0$ for Ω_2.

Remark. We also note that for the domain

$$\Omega = \{r(z) < 0 \colon r(z) = 2\,\mathrm{Re}\, z_3 + |z_2|^4\}$$

there is a subelliptic estimate at $z = 0$, but the domains

$$\Omega_m = \{r(z) < 0 \colon r(z) \colon 2\,\mathrm{Re}\, z_3 - |z_1^m z_2|^2 + |z_2|^4\}$$

(which are small perturbations of Ω) there is no subelliptic estimate at $z = 0$.

3. Exact estimate for $(n-1)$ forms in some domains

Kohn showed in [6] that if a domain Ω is pseudoconvex and $x_0 \in b\Omega$ is of $\mathrm{reg}\, O^{n-1}(x_0) = m$, then an exact estimate of $\varepsilon = \frac{1}{m}$ holds at x_0 for $n-1$ forms. Catlin [2] developed a technique by using Hörmander's estimate [5] with weight function to give sufficient conditions for subelliptic estimates.

Consider the nonpseudoconvex domain defined by

$$r = 2\,\mathrm{Re}\, z_3 - |z_1|^2 + |z_2|^4.$$

In [4] we showed that a subelliptic estimate holds at $z_0 = 0$ for 2-forms, and using Proposition 3.2 of [4] (or Theorem 2.1 of this paper) we know that $\varepsilon \leq \frac{1}{4}$.

Adapting Catlin's technique in [2] to nonpseudoconvex domains we show in the following theorem that we actually have a subelliptic estimate of order $\varepsilon = \frac{1}{4}$ at $z = 0$ for this domain.

Theorem 3.1. *Suppose $\Omega = \{z : r(z) < 0\}$ and that near the origin there exist a smooth nonvanishing tangential vector field L and a real smooth function φ on $b\Omega$ such that $L\varphi(0) \neq 0$ and*

$$\partial\overline{\partial}r(L, \overline{L}) \geq C|\varphi(z)|^m$$

for $z \in b\Omega$ for some $C > 0$ and some even integer m. Then a subelliptic estimate of order $\varepsilon = \frac{1}{m+2}$ holds for $n-1$ forms near the origin.

To prove the theorem, we need an integration by parts lemma which involves a weight function. Let φ be a real C^2-function on Ω, and u a $(0, n-1)$ form in Ω, then we define

$$\|u\|_\varphi^2 = \int_\Omega |u|^2 e^{-\varphi}\, dV.$$

Let $L_1, \ldots, L_n$ be C^∞ vector fields with values in $T^{1,0}$, and $L_1, \ldots, L_{n-1}$ be tangential to $b\Omega$.

Lemma 3.2. *Let Ω be a domain in $\mathbb{C}^n$ with smooth boundary, $L \in T^{1,0}(b\Omega)$, $u \in C_0^\infty(\overline{\Omega})$ and $\varphi \in C^2(\overline{\Omega})$. Then*

$$\begin{aligned}\|Lu\|_\varphi^2 = \|\overline{L}u\|_\varphi^2 &+ \int_{b\Omega} \lambda|u|^2 e^{-\varphi}\, dS + \int_\Omega (\overline{L}L\varphi)|u|^2 e^{-\varphi}\, dV \\ &- \int_\Omega |L\varphi|^2|u|^2 e^{-\varphi}\, dV + 2\,\mathrm{Re}\langle (Lu)(\overline{L}\varphi)e^{-\varphi}, u\rangle + \langle (fu)(L\varphi), ue^{-\varphi}\rangle \\ &+ \langle g(Lu)e^{-\varphi}, u\rangle + \langle h(Lu)e^{-\varphi}, u\rangle + \langle (Du)e^{-\varphi}, u\rangle\end{aligned}$$

where $\lambda = \langle \partial\overline{\partial}r, L \wedge \overline{L}\rangle$, $f, g, h \in C^\infty(\overline{\Omega})$ and

$$D = \sum_{i=1}^{n} a_i \overline{L}_i + \sum_{j=1}^{n-1} b_j L_j$$

where the a_i's and b_j's are smooth functions.

Proof. We will repeatedly use the formula

$$\langle Lu, v\rangle = -\langle u, \overline{L}v\rangle + \langle u, gv\rangle, \qquad L \in T^{1,0}(b\Omega),$$

where g is smooth in $U \cap \overline{\Omega}$.

$$\begin{aligned}
\|Lu\|_\varphi^2 &= \langle (Lu)e^{-\varphi}, Lu\rangle \\
&= -\langle \overline{L}((Lu)e^{-\varphi}), u\rangle + \langle g(Lu)e^{-\varphi}, u\rangle \\
&= -\langle (\overline{L}Lu)e^{-\varphi}, u\rangle - \langle (Lu)(\overline{L}e^{-\varphi}), u\rangle + \langle g(Lu)e^{-\varphi}, u\rangle \\
&= -\langle ([\overline{L}, L]u)e^{-\varphi}, u\rangle - \langle (L\overline{L}u)e^{-\varphi}, u\rangle \\
&\quad + \langle (Lu)(\overline{L}\varphi)e^{-\varphi}, u\rangle + \langle g(Lu)e^{-\varphi}, u\rangle \\
&= \int_{b\Omega} \lambda |u|^2 e^{-\varphi}\, dS + \langle (Du)e^{-\varphi}, u\rangle - \langle L((\overline{L}u)e^{-\varphi}), u\rangle \\
&\quad -\langle (\overline{L}u)(L\varphi)e^{-\varphi}, u\rangle + \langle (Lu)(\overline{L}\varphi)e^{-\varphi}, u\rangle + \langle g(Lu)e^{-\varphi}, u\rangle \\
&= \int_{b\Omega} \lambda |u|^2 e^{-\varphi}\, dS + \langle (\overline{L}u)e^{-\varphi}, \overline{L}u\rangle + \langle h(\overline{L}u)e^{-\varphi}, u\rangle - \langle (\overline{L}u)(L\varphi)e^{-\varphi}, u\rangle \\
&\quad + \langle (Lu)(\overline{L}\varphi)e^{-\varphi}, u\rangle + \langle g(Lu)e^{\varphi}, u\rangle + \langle (Du)e^{-\varphi}, u\rangle \\
&= \|\overline{L}u\|_\varphi^2 + \int_{b\Omega} \lambda |u|^2 e^{-\varphi}\, dS - \langle \overline{L}(uL\varphi), ue^{-\varphi}\rangle + \langle u\overline{L}L\varphi, ue^{-\varphi}\rangle \\
&\quad + \langle (Lu)(\overline{L}\varphi)e^{-\varphi}, u\rangle + \langle g(Lu)e^{-\varphi}, u\rangle + \langle h(\overline{L}u)e^{-\varphi}, u\rangle + \langle (Du)e^{-\varphi}, u\rangle \\
&= \|\overline{L}u\|_\varphi^2 + \int_{b\Omega} \lambda |u|^2 e^{-\varphi}\, dS + \langle uL\varphi, L(ue^{-\varphi})\rangle + \langle (fu)(L\varphi), ue^{-\varphi}\rangle \\
&\quad + \langle u\overline{L}L\varphi, ue^{-\varphi}\rangle + \langle (Lu)(\overline{L}\varphi)e^{-\varphi}, u\rangle + \langle g(Lu)e^{-\varphi}, u\rangle \\
&\quad + \langle h(\overline{L}u)e^{-\varphi}, u\rangle + \langle (Du)e^{-\varphi}, u\rangle \\
&= \|\overline{L}u\|_\varphi^2 + \int_{b\Omega} \lambda |u|^2 e^{-\varphi}\, dS - \langle uL\varphi, u(L\varphi)e^{-\varphi}\rangle + \langle uL\varphi, (Lu)e^{-\varphi}\rangle \\
&\quad + \langle u\overline{L}L\varphi, ue^{-\varphi}\rangle + \langle (Lu)(\overline{L}\varphi)e^{-\varphi}, u\rangle + \langle (fu)(L\varphi), ue^{-\varphi}\rangle \\
&\quad + \langle g(Lu)e^{-\varphi}, u\rangle + \langle h(\overline{L}u)e^{-\varphi}, u\rangle + \langle (Du)e^{-\varphi}, u\rangle \\
&= \|\overline{L}u\|_\varphi^2 + \int_{b\Omega} \lambda |u|^2 e^{-\varphi}\, dS + \int_\Omega (\overline{L}L\varphi)|u|^2 e^{-\varphi}\, dV - \int_\Omega |L\varphi|^2 |u|^2 e^{-\varphi}\, dV \\
&\quad + 2\operatorname{Re}\langle (Lu)(\overline{L}\varphi)e^{-\varphi}, u\rangle + \langle (fu)(L\varphi), ue^{-\varphi}\rangle + \langle g(Lu)e^{-\varphi}, u\rangle \\
&\quad + \langle h(\overline{L}u)e^{-\varphi}, u\rangle + \langle (Du)e^{-\varphi}, u\rangle.
\end{aligned}$$

From Lemma 3.2, we easily get

Corollary 3.3. *With the same notations as in Lemma* 3.2, *given* $\varepsilon > 0$, *there exists a neighborhood* U *of* $x_0 \in b\Omega$ *such that when* $u \in C_0^\infty(U \cap \overline{\Omega})$, *we have*

$$\begin{aligned}
2\|Lu\|_\varphi^2 \geq \frac{1}{2}\|\overline{L}u\|_\varphi^2 &+ \int_{b\Omega} \lambda |u|^2 e^{-\varphi}\, dS + \int_\Omega (\overline{L}L\varphi)|u|^2 e^{-\varphi}\, dV \\
&- 3\int_\Omega |L\varphi|^2 |u|^2 e^{-\varphi}\, dV - \varepsilon\left(\sum_{i=1}^{n-1} \|L_i u\|_\varphi^2 + \|\overline{L}_n u\|_\varphi^2\right) - O(\|u\|_\varphi)^2.
\end{aligned}$$

Proof of Theorem 3.1. We choose vector fields $L_1, \ldots, L_{n-1}, L_n$ with L_i tangential, $i = 1, 2, \ldots, n-1$ and $L_1 = L$ in the theorem. Let $\omega_1, \ldots, \omega_n$ be $(1, 0)$ forms dual to $L_1, \ldots, L_n$ and let $u \in D_U^{0,n-1}$ where the size of U will be chosen later. We may assume that

$$u = u_{12\cdots n-1}\overline{\omega}_1 \wedge \overline{\omega}_2 \wedge \cdots \wedge \overline{\omega}_{n-1}.$$

Again we will use u to denote the $n-1$ form and the function $u_{12\cdots n-1}$. Clearly

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 \approx \|L_1 u\|^2 + \|L_2 u\|^2 + \cdots + \|L_{n-1} u\|^2 + \|\overline{L}_n u\|^2$$

and

$$\|L_i u\|_\varphi^2 \approx \|L_i u\|^2$$

if φ is both bounded above and below.

If $\varphi(0) \neq 0$, clearly a subelliptic estimate of order $\varepsilon = \frac{1}{2}$ holds at the origin and conclusion follows. With a change of coordinates we may assume that $\varphi(z) = x_1$. For the sake of convenience we let $m = 2p - 2$.

By Corollary 3.3 there exists a neighborhood U of $x_0 = 0$ such that when $u \in C_0^\infty(U \cap \overline{\Omega})$

$$\begin{aligned} 2\|L_1 u\|_\varphi^2 \geq{}& \frac{1}{2}\|\overline{L}_1 u\|_\varphi^2 + \text{const}\int_{b\Omega} |x_1|^{2p-2}|u|^2 e^{-\varphi}\, dS \\ &+ \int_\Omega (\overline{L}_1 L_1 \varphi)|u|^2 e^{-\varphi}\, dV - 3\int_\Omega |L_1\varphi|^2 |u|^2 e^{-\varphi}\, dV \\ &- \varepsilon(\|L_1 u\|_\varphi^2 + \cdots + \|L_{n-1} u\|_\varphi^2 + \|\overline{L}_n u\|_\varphi^2) - O(\|u\|_\varphi^2). \end{aligned} \tag{3.1}$$

Let $\psi = c\mu(2^{k/p}|x_1|^2)$ where $\mu \in C^\infty(\mathbb{R})$ satisfies

$$\mu(x) = \begin{cases} x, & 0 \leq x \leq \frac{3}{4}, \\ 0, & x \geq 2, \end{cases}$$

$$|\mu'(x)| \leq N \quad \text{and} \quad |\mu''(x)| \leq N \quad \text{for some } N > 0,$$

and c is some positive number to be chosen later.

If we define $\varphi = \frac{1}{M}e^\psi$ where M is a constant so that $M \geq 3e^{\psi(z)}$ for all z, then φ is clearly bounded above and below. Now

$$\overline{L}_1 L_1 \varphi = \frac{1}{M}((\overline{L}_1 L_1 \psi)e^\psi + |L_1\psi|^2 e^\psi),$$

$$|L_1\varphi|^2 = \frac{1}{M^2}|L_1\psi|^2 e^{2\psi},$$

and

$$\begin{aligned} \overline{L}_1 L_1 \psi ={}& c2^{k/p+1}(|L_1 x_1|^2 + x_1 \overline{L}_1 L_1 x_1)\mu'(2^{k/p}|x_1|^2) \\ &+ c2^{2k/p+2}|x_1|^2 |L_1 x_1|^2 \mu''(2^{k/p}|x_1|^2). \end{aligned}$$

When the neighborhood U is small enough, we have

$$\overline{L}_1 L_1 \psi \geq c_1(2^{k/p}\mu'(2^{k/p}|x_1|^2) + 2^{2k/p}|x_1|^2 \mu''(2^{k/p}|x_1|^2)).$$

Then (3.1) gives

$$
\begin{aligned}
(3.2)\quad 2\|L_1 u\|_\varphi^2 &\geq \frac{1}{2}\|\overline{L}_1 u\|_\varphi^2 + \text{const}\int_{b\Omega} |x_1|^{2p-2}|u|^2 e^{-\varphi}\, dS \\
&\quad + \frac{1}{M}\int_\Omega (\overline{L}_1 L_1 \psi)|u|^2 e^{\psi} e^{-\varphi}\, dV \\
&\quad - \varepsilon(\|L_1 u\|_\varphi^2 + \cdots + \|L_{n-1}u\|_\varphi^2 + \|\overline{L}_n u\|_\varphi^2) - O(\|u\|_\varphi^2) \\
&\geq \text{const}\left(\|\overline{L}_1 u\|^2 + \int_{b\Omega} |x_1|^{2p-2}|u|^2\, dS + \int_\Omega (\overline{L}_1 L_1 \psi)|u|^2\, dV\right) \\
&\quad - \varepsilon(\|L_1 u\|^2 + \cdots + \|L_{n-1}u\|^2 + \|\overline{L}_n u\|^2) - O(\|u\|^2) \\
&\geq \text{const}\left(\|\overline{L}_1 u\|^2 + \int_{b\Omega} |x_1|^{2p-2}|u|^2\, dS \right.\\
&\qquad\qquad \left. + c_1 \int_\Omega (2^{k/p}\mu' + 2^{2k/p}|x_1|^2 \mu'')|u|^2\, dV\right) \\
&\quad - \varepsilon(\|L_1 u\|^2 + \cdots + \|L_{n-1}u\|^2 + \|\overline{L}_n u\|^2) - O(\|u\|^2).
\end{aligned}
$$

Let the tangential Fourier transform variable be $\xi = (\xi_1, \xi_2, \ldots, \xi_{2n-1})$. Assuming $\hat{u}(\xi)$ is supported in $2^{k-2} \leq |\xi| \leq 2^k$, then

$$
\begin{aligned}
\int_{b\Omega} |x_1|^{2p-2}|u|^2\, dS &= \int_{b\Omega} |\widehat{x_1^{p-1}u}|^2\, dS \\
&\geq \int_\Omega (1 + |\xi|^2)^{1/2} |\widehat{x_1^{p-1}u}|^2\, dV - \|L_1 u\|^2 - \cdots - \|L_{n-1}u\|^2 - \|\overline{L}_n u\|^2 - \|u\|^2 \\
&\geq \text{const}\, 2^k \int_\Omega |x_1^{p-1}u|^2\, dV - \|L_1 u\|^2 - \cdots - \|L_{n-1}u\|^2 - \|\overline{L}_n u\|^2 - \|u\|^2.
\end{aligned}
$$

Putting this into (3.2), we get

(3.3)

$$
\begin{aligned}
\|L_1 u\|^2 &\geq \text{const}\left(\|\overline{L}_1 u\|^2 + \int_\Omega (2^k |x_1|^{2p-2} + c_1 2^{k/p}\mu' + c_1 2^{2k/p}|x_1|^2\mu'')|u|^2\, dV\right) \\
&\quad - (\|L_1 u\|^2 + \cdots + \|L_{n-1}u\|^2 + \|\overline{L}_n u\|^2 + \|u\|^2).
\end{aligned}
$$

Consider the function

$$f(z_1) = 2^k|x_1|^{2p-2} + c_1 2^{k/p}\mu'(2^{k/p}|x_1|^2) + c_1 2^{2k/p}|x_1|^2\mu''(2^{k/p}|x_1|^2).$$

Case 1. $|x_1|^2 \leq \frac{1}{4}2^{-k/p}$.

In this case $\mu' = 1$ and $\mu'' = 0$, hence $f(x_1) \geq \text{const}\, 2^{k/p}$.

Case 2. $\frac{1}{4}2^{-k/p} \leq |x_1|^2 \leq 2.2^{-k/p}$.

In this case $|c_1 2^{k/p}\mu' + c_1 2^{2k/p}|x_1|^2\mu''| \leq 3c_1 N 2^{k/p}$. If we choose c so that $c_1 = \frac{1}{6N}2^{2-2p}$, then $f(x_1) \geq \text{const}\, 2^{k/p}$.

Case 3. $|x_1|^2 \geq 2.2^{-k/p}$.

In this case $\mu' = \mu'' = 0$, hence $f(x_1) \geq \text{const}\, 2^{k/p}$.

We conclude that $f(x_1) \geq \text{const}\, 2^{k/p}$ for all x_1. Putting this into (3.3) we have when $\operatorname{supp}\hat{u}$ is in $2^{k-2} \leq |\xi| \leq 2^k$,

$$\|L_1 u\|^2 \geq \text{const}\left(\|\overline{L}_1 u\|^2 + \int_\Omega 2^{k/p}|u|^2\, dV - (\|L_1 u\|^2 + \cdots + \|L_{n-1}u\|^2 + \|\overline{L}_n u\|^2 + \|u\|)^2\right). \tag{3.4}$$

Define a function $\chi_0 \in C^\infty(\mathbb{R})$ which satisfies

(i) $$0 \leq \chi_0 \leq 1,$$

(ii) $$\chi_0(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2}, \\ 0, & x \geq 1, \end{cases}$$

and define χ_k, $k = 1, 2, \ldots$, as follows:

$$\chi_k(x) = \chi_0\left(\frac{|x|}{2^k}\right) - \chi_0\left(\frac{|x|}{2^{k-1}}\right).$$

Then $\sum_{k=0}^\infty \chi_k = 1$ and the operator $\zeta_k u = \mathscr{F}^{-1}(\chi_k \hat{u})$ is a pseudodifferential operator of order 0, uniformly in k. Clearly $\widehat{\zeta_k u}$ is supported in $2^{k-2} \leq |\xi| \leq 2^k$. Hence

$$\begin{aligned}
|||u|||_{1/2p}^2 &= \iint (1 + |\xi|^2)^{1/2p} |\hat{u}(\xi, r)|^2\, d\xi\, dr \\
&\leq \text{const} \sum_{k=0}^\infty \iint (1 + |\xi|^2)^{1/2p} |\widehat{\zeta_k u}(\xi, r)|^2\, d\xi\, dr \\
&\leq \text{const} \sum_{k=0}^\infty 2^{k/p} \int_\Omega |\zeta_k u(x, r)|^2\, dV \\
&\leq \text{const} \sum_{k=0}^\infty (\|L_1(\zeta_k u)\|^2 + \|L_2(\zeta_k u)\|^2 + \cdots \\
&\qquad + \|L_{n-1}(\zeta_k u)\|^2 + \|\overline{L}_n(\zeta_k u)\|^2 + \|\zeta_k u\|^2) \\
&\leq \text{const} \sum_{k=0}^\infty (\|\overline{\partial}(\zeta_k u)\|^2 + \|\overline{\partial}^*(\zeta_k u)\|^2 + \|\zeta_k u\|^2) \\
&\leq \text{const} \sum_{k=0}^\infty (\|\zeta_k \overline{\partial} u\|^2 + \|\zeta_k \overline{\partial}^* u\|^2 + \|\zeta_k u\|^2) \\
&\leq \text{const}(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 + \|u\|^2)
\end{aligned}$$

where we used (3.4) in line 4 and the following lemma in line 6.

Lemma 3.4. *Let* $L = \sum a_i \partial/\partial x_i$ *where* a_i *are* C^∞ *functions and* ζ_k *be the same as in Theorem* 2.1, *then*

$$\sum_{k=0}^{\infty} \|[L, \zeta_k]u\|^2 \leq \text{const}\,\|u\|^2.$$

We refer the reader to Lemma 2.5 of [2] for the proof of this lemma.

Corollary 3.4. *Let* Ω *be a domain in* $\mathbb{C}^n$ *defined by*

$$\{r(z) < 0 : r(z) = 2\,\mathrm{Re}\, z_n + \varphi(z_2, \ldots, z_{n-2}) + |z_1|^{2p}\}$$

where φ *is real and* $\varphi \in C^\infty(\mathbb{C}^{n-2})$. *Then a subelliptic estimate of order* $\frac{1}{2p}$ *holds at* $z = 0$.

Proof. The vector field $L = r_{z_n} \partial/\partial z_1 - r_{z_1} \partial/\partial z_n$ satisfies

$$\partial\bar{\partial} r(L, \bar{L}) \geq C|x_1|^{2p-2}.$$

We get the conclusion by appling Theorem 3.1.

References

1. D. Catlin, *Necessary conditions for subellipticity of the* $\bar{\partial}$*-Neumann problem*, Ann. of Math. (2) **117** (1983), 147–141.
2. ——, *Subelliptic estimates for the* $\bar{\partial}$*-Neumann problem on pseudoconvex domains*, Ann. of Math. (2) **126** (1987), 131–191.
3. M. Derridj, *Inégalités a priori et estimation sous-elliptique pour* $\bar{\partial}$ *dan ouverts non pseudo-convexes*, Math. Ann. **249** (1980), 27–48.
4. L. Ho, *Subellipticity of the* $\bar{\partial}$*-Neumann problem on nonpseudoconvex domains*, Trans. Amer. Math. Soc. **291** (1985), 43–73.
5. L. Hörmander, L^2 *estimates and existence theorems for the* $\bar{\partial}$*-operator*, Acta Math. **113** (1965), 89–152.
6. J. J. Kohn, *Subellipticity of the* $\bar{\partial}$*-Neumann problem on pseudoconvex domains, sufficient conditions*, Acta Math. **142** (1979), 79–122.

Department of Mathematics, Hong Kong Polytechnic, Hung Hom, Hong Kong

Current address: Department of Mathematics, The Wichita State University, Wichita, Kansas 67208